Fourier coefficients for Continuous signal $x(t) \to a_k$

Asking deriving coefficients comes with periodic signal.

(CT FS) Basic concept of continuous Fourier coefficients

$x(t)$

$$x(t) : Periodic\ signal$$
$$T : Fundamental\ Period$$
$$\omega_0 = \frac{2\pi}{T} \quad \& \quad f_0 = \frac{1}{T}(freq)$$
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

(CT FS) Continuous-Time, Fourier Series

$x(t) \xrightarrow{FS} a_k$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

or

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

(CT IFS) Continuous-Time, Inverse Fourier Series

$a_k \xrightarrow{IFS} x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

Properties of Continuous-Time Fourier Series

Property	Periodic Signal $x(t)$	Fourier Series Coefficients a_k
Linearity	Ax(t)+By(t)	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (Periodic with period T/α)	a_{-k}
Periodic convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=\langle N\rangle} a_l b_{k-1}$
Different-iation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$
Real & Even Signal	$x[n]$ real and even	a_k real and even
Real & Odd Signal	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signal	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\}[x(t)\ real] \\ x_o(t) = \theta d\{x(t)\}[x(t)\ real] \end{cases}$	Re $\{a_k\}$ j Im$\{a_k\}$

Fourier coefficients for Discrete signal $x[n] \to a_k$

Asking deriving coefficients comes with periodic signal.

(DT FS) Basic concept of discrete Fourier coefficients

$x[n]$

$$x[n] : Periodic\ signal$$
$$N : Fundamental\ Period \text{ (LCM of } 2\pi)$$
$$\omega_0 = \frac{2\pi}{N} \quad \& \quad f_0 = \frac{1}{T}(freq)$$
$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

(DT FS) Discrete-Time, Fourier Series

$x[n] \xrightarrow{FS} a_k$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$$

or

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$$

(DT IFS) Discrete-Time, Inverse Fourier Series

$a_k \xrightarrow{IFS} x[n]$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

Properties of Discrete-Time Fourier Series

Property	Periodic Signal $x[n]$	Fourier Series Coefficients a_k
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple} \end{cases}$	$\frac{1}{m}a_k$ (viewed as periodic with period mN)
Periodic convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-1}$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$
Real & Even Signal	$x[n]$ real and even	a_k real and even
Real & Odd Signal	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signal	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\}[x[n]\ real] \\ x_o[n] = \theta d\{x[n]\}[x[n]\ real] \end{cases}$	Re $\{a_k\}$ j Im$\{a_k\}$

Convert Trigonometric to exponential form

Many cases require to convert Trigonometric to $e^{j\theta}$ form.

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\tan(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j\,(e^{j\theta} + e^{-j\theta})}$$

Fourier transform for <u>Continuous</u>-time signal $\boldsymbol{x(t)}$

Most of case, aperiodic signals comes…

(CT FT) Continuous-Time, Fourier Transform (periodic)

$\tilde{x}(t)$: *single sliced periodic sig*

$$x(t) \xrightarrow{FT} X(j\omega) \qquad a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt$$

$$X(j\omega) = T\, a_k$$

(CT FT) Continuous-Time, Fourier Transform (aperiodic)

$$x(t) \xrightarrow{FT} X(j\omega) \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

(CT IFT) Continuous-Time, Inverse Fourier Transform

$$X(jw) \xrightarrow{IFT} x(t) \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

Properties of <u>Continuous</u> Fourier <u>Transform</u>

Property	Periodic Signal $\boldsymbol{x(t)}$	Fourier Transform $\boldsymbol{X(j\omega)}$
Linearity	$a\,x(t) + b\,y(t)$	$a\,X(j\omega) + b\,Y(j\omega)$
Time Shifting	$x(\mathrm{t} - \mathrm{t}_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X\big(j(\omega - \omega_0)\big)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Freq Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multipli-cation	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}[\,X(j\theta)\, Y\big(j(\omega-\theta)\big)\,]\,d\theta$
Differentiation in time	$\frac{d}{dt}\,x(t)$	$j\omega\, X(j\omega)$
Integration	$\int_{-x}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
symmetry for Real & Even	$x(t)$ *real and even*	$X(j\omega)$ *real and even*
symmetry for Real & Odd	$x(t)$ *real and odd*	$X(j\omega)$ *pure imaginary, and odd*
Even-Odd decomposition for Real-Signal	$x_e(t) = Ev\{x(t)\}$ $x_o(t) = Od\{x(t)\}$ $[x(t)$ real $]$	Re $\{\,X(j\omega)\,\}$ j Im$\{\,X(j\omega)\,\}$
Parseval's Relation	$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$	

Basic <u>Continuous</u> Fourier <u>Transform Pairs</u>

Signal $\boldsymbol{x(t)}$	Fourier Transform $\boldsymbol{X(j\omega)}$ Fourier Series Coefficients $\boldsymbol{a_k}$ (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$X(j\omega) = 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta(\omega - k\omega_0)$ $a_k{=}a_k$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$ $a_1 = 1$ $a_k = 0$, otherwise
$\cos\,\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ $a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\,\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ $a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$ $a_0 = 1, a_k = 0, k \neq 0$
Periodic square wave : $x(t) = \begin{cases}1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2}\end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin\,k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$ $\frac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin\,k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty}\delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$ $a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases}1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1\end{cases}$	$\frac{2\sin\,\omega T_1}{\omega}$
$\frac{\sin\,Wt}{\pi t}$	$X(j\omega) = \begin{cases}1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W\end{cases}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), Re\{a\} > 0$	$\frac{1}{a + j\omega}$
$t\,e^{-at}u(t), Re\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$ $Re\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$

(1칸에 2칸이 있는 경우, 아래칸 Coefficients 에 대한 기술)

Fourier transform for <u>Discrete</u>-time signal $\boldsymbol{x[n]}$

Most of case, aperiodic signals comes…

(DT FT) Discrete-Time, Fourier Transform

$$x[n] \xrightarrow{FT} X(e^{j\omega}) \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

(DT IFT) Discrete-Time, Inverse Fourier Transform

$$X(e^{j\omega}) \xrightarrow{IFT} x[n] \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Properties of <u>Discrete</u> Fourier <u>Transform</u>

Property	Periodic Signal $\boldsymbol{x[n]}$	Fourier Transform $\boldsymbol{X(e^{j\omega})}$
	$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ *periodic with period* $\boldsymbol{2\pi}$
Linearity	$a\,x[n] + b\,y[n]$	$a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X\left(e^{j(\omega-\omega_0)}\right)$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = k \\ 0, & \text{if } n \neq k \end{cases}$ $n =$ *multiple of k* $n \neq$ *multiple of k*	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multipli-cation	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y\left(e^{j(\omega-\theta)}\right)d\theta$
Differencing in time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumul-ation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
Differentiation in Freq	$n\,x[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
symmetry for Real & Even	$x[n]$ *real and even*	$X(e^{j\omega})$ *real and even*
symmetry for Real & Odd	$x[n]$ *real and odd*	$X(e^{j\omega})$ *pure imaginary, and odd*
Even-Odd decomposition for Real-Signal	$x_e[n] = Ev\{x[n]\}$ $x_o[n] = Od\{x[n]\}$ $[x[n]$ real $]$	Re $\{X(e^{j\omega})\}$ j Im$\{X(e^{j\omega})\}$
Parseval's Relation	$\sum_{n=-\infty}^{+\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$	

Basic <u>Discrete</u> Fourier <u>Transform Pairs</u>

Signal $\boldsymbol{x(t)}$	Fourier Transform $\boldsymbol{X(e^{j\omega})}$ Fourier Series Coefficients $\boldsymbol{a_k}$ (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$X(e^{j\omega}) = 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$ $a_k = a_k$
$e^{j\omega_0 n}$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - \omega_0 - 2\pi l)$
$\cos\,\omega_0 n$	$\pi\sum_{l=-\infty}^{+\infty}\{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$ $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ $\frac{\omega_0}{2\pi}$ = irrational, The signal is aperiodic
$\sin\,\omega_0 n$	$\frac{\pi}{j}\sum_{l=-\infty}^{+\infty}\{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$ $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - 2\pi l)$ $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave : $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$ $a_k = \frac{\sin\left[(2\pi k/N)\left(N_1 + \frac{1}{2}\right)\right]}{N\sin[2\pi k/2N]}, k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty}\delta[n - kN]$	$\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{N}\right)$ $a_k = \frac{1}{N}$ for all k
$a^n u[n] \quad, \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$x[n]\begin{cases} 1, & \lvert n\rvert < N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin\left[\omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin(\omega/2)}$
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\text{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π
$\delta[n]$	1
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty}\pi\delta(\omega - 2\pi k)$
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$(n+1)a^n u[n]$ $, \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{(n+r-1)!}{n!\,(r-1)!}a^n u[n]$ $, \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$

(1칸에 2칸이 있는 경우, 아래칸 Coefficients 에 대한 기술)